Anisotropy

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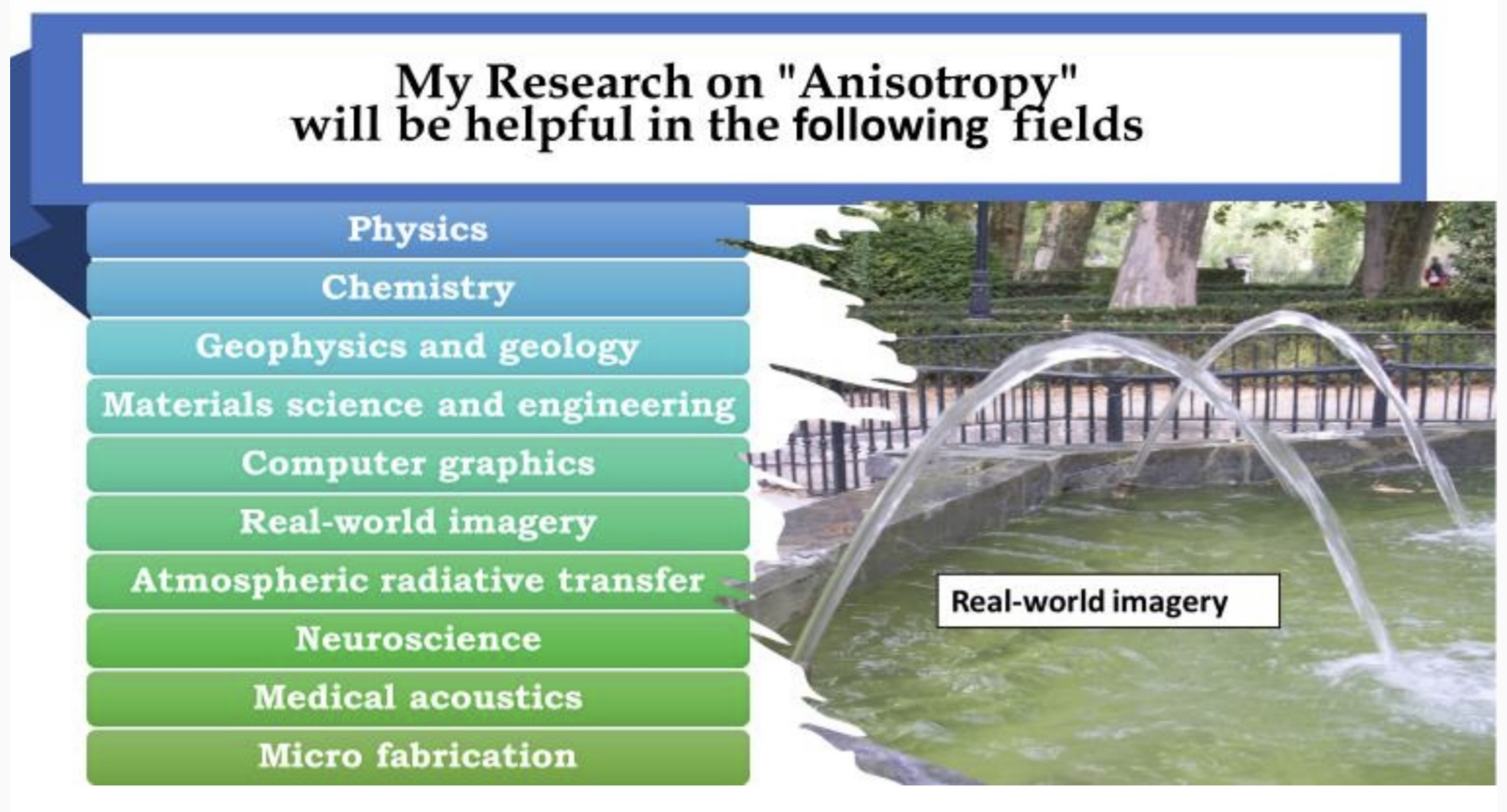

Finsler norm

Let $H:\mathbb{R}^N\to\mathbb{R}$ be a nonnegative, convex function of class $C^2(\mathbb{R}^N\setminus\{0\})$, which satisfies

$$\begin{cases} H(\xi)\ge 0,\, H(\xi)=0 \iff \xi=0\\ H(t\xi)=|t|H(\xi), \quad \forall\xi\in\mathbb{R}^N,\ \forall t\in\mathbb{R}\\ H(\xi+\eta)\le H(\xi)+H(\eta).\end{cases}$$

H is called a Finsler norm.

The polar function of H is the function $H^0:\mathbb{R}^N\to\mathbb{R}$ defined by

$$H^0(x)=\sup_{\xi\in\mathbb{R}^N\setminus\{0\}}\frac{\xi\cdot x}{H(\xi)} \qquad x\in\mathbb{R}^N.$$

H^0 is also a norm on $\mathbb{R}^N$.

Asymptotic behavior of least energy solutions to the Finsler Lane-Emden problem with large exponents

Finsler Lane-Emden problem

$$(E_p)\quad \begin{cases} -Q_N u=u^p & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $p>1$ is any positive number and Q_N is the Finsler N-Laplace operator defined by

Finsler N-Laplacian

$$Q_N u(x)=\operatorname{div}\left(H(\nabla u(x))^{N-1}(\nabla_\xi H)(\nabla u(x))\right)$$

where $H=H(\xi)$ is any norm on $\mathbb{R}^N$ (Finsler norm).

Our aim is to extend the results of Ren and Wei (1994, 1995, 1996) to the anisotropic problem (E_p).

Our first result is the following L^∞-bound of least energy solutions.

Theorem 1 (DCDS, 42, no.10, (2022), pp.5063–5086)

Let u_p be a least energy solution to (E_p). Then there exist C_1, C_2 (independent of p), such that

$$0<C_1\le\|u_p\|_{L^\infty(\Omega)}\le C_2<\infty \quad \text{for } p \text{ large enough.}$$

Furthermore, we have

$$\lim_{p\to\infty}p^{N-1}\int_\Omega H(\nabla u_p)^N dx=\lim_{p\to\infty}p^{N-1}\int_\Omega u_p^{p+1}dx=\left(\frac{Ne\beta_N}{N-1}\right)^{N-1}$$

where $\beta_N=N(N\kappa_N)^{\frac{1}{N-1}}$, $\kappa_N=|\mathcal{W}|$ is the volume of the unit Wulff ball $\mathcal{W}=\{x\in\mathbb{R}^N: H^0(x)<1\}$.

On the asymptotic behavior of the L^∞-norm of u_p, we have

Theorem 2 (ibid.)

Let u_p be a least energy solution to (E_p). Then it holds that

$$1\le\limsup_{p\to\infty}\|u_p\|_{L^\infty(\Omega)}\le e^{\frac{N-1}{N}}.$$

The estimate from above is new even for the case $Q_N=\Delta_N$.

Conjecture

Least energy solution u_p must satisfy

$$\limsup_{p\to\infty}\|u_p\|_{L^\infty(\Omega)}=e^{\frac{N-1}{N}}.$$

This conjecture is known to be true when $N=2$ and $H(\xi)=|\xi|$.

Applications of p-harmonic transplantation for functional inequalities involving a Finsler norm

A transformation between symmetric functions:

Let $u=u(x)$ be a radially symmetric function on $\mathbb{R}^N$, i.e., there exists a function U defined on $[0,+\infty)$ s.t. $u(x)=U(|x|)$. Also let $v=v(y)$ be a Finsler radially symmetric function on $\mathcal{W}_R$ of the form $v(y)=V(H^0(y))$ for some $V=V(s)$, $s\in[0,R)$, where $R>0$ be any number. Fix $1<p<N$ and assume that u and v are related with each other by the transformation

$$\begin{cases} r=|x|, \quad x\in\mathbb{R}^N,\\ s=H^0(y), \quad y\in\mathcal{W}_R\subset\mathbb{R}^N,\\ r^{\frac{p-N}{p-1}}=s^{\frac{p-N}{p-1}}-R^{\frac{p-N}{p-1}},\\ u(x)=U(r)=V(s)=v(y).\end{cases}$$

This kind of transformation (p-harmonic transplantation) originates from J. Hersh (1969).

From $\mathbb{R}^N$ to $\mathcal{W}_R$

If we have some functional inequalities for radially symmetric functions on $\mathbb{R}^N$, then we have new inequalities with a Finsler norm for Finsler symmetric functions on $\mathcal{W}_R$. (Proposition 1).

From B_R to $\mathbb{R}^N$

If we have some functional inequalities for radially symmetric functions on B_R, then we have new inequalities with a Finsler norm for Finsler symmetric functions on $\mathbb{R}^N$. (Proposition 2).

Theorem 1 (The sharp L^p-Sobolev inequality on $\mathcal{W}_R$)

Let $N\ge 2$, $1<p<N$ and $p^*=\frac{Np}{N-p}$. Then for any Finsler radially symmetric function $v\in W_0^{1,p}(\mathcal{W}_R)$, the inequality

$$\tilde{S}_{N,p}\left(\int_{\mathcal{W}_R}\frac{|v(y)|^{p^*}}{\left(1-\left(\frac{H^0(y)}{R}\right)^{\frac{N-p}{p-1}}\right)^{\frac{p(N-1)}{N-p}}}dy\right)^{p/p^*}\le\int_{\mathcal{W}_R}H(\nabla v(y))^p dy$$

holds true. Here

$$\tilde{S}_{N,p}=S_{N,p}\left(\frac{\omega_{N-1}}{N\kappa_N}\right)^{p/p^*-1}.$$

The equality holds iff $v(y)=V(H^0(y))$, where

$$V(H^0(y))=\left(a+b\left((H^0(y))^{\frac{p-N}{p-1}}-R^{\frac{p-N}{p-1}}\right)^{\frac{p}{p-N}}\right)^{\frac{p-N}{p}}$$

for some $a,b>0$.

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